ON THE MAXIMUM PARTIAL SUMS OF SEQUENCES OF INDEPENDENT RANDOM VARIABLES(1)

BY

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1. **Introduction.** In this paper we deal with a sequence of independent random variables X_n, $n=1, 2, \cdots$. We write

$$(1) \qquad S_n = \sum_{\nu=1}^{n} X_\nu,$$

$$(2) \qquad S_n^* = \max_{1 \leq \nu \leq n} |S_\nu|.$$

Two types of fundamental limit theorems are known about S_n, the one clustering around the central limit theorem and the other the law of the iterated logarithm.

In 1945 Feller [12](2) called attention to the study of the behavior of S_n^*. Since then an important result has been obtained by Erdös and Kac [8], namely, the limiting distribution of S_n^* for sufficiently general sequences of X_n. This corresponds to the central limit theorem for S_n. Now under certain conditions when the distribution of S_n tends to the normal distribution, an estimate of the difference of the two distributions has been given by Liapounoff [17], Cramér [5], Berry [3] and Essen [9]. Cramér [6] and Feller [10] have also obtained more precise estimates for this difference for certain domains of variation of S_n, which proved essential to the general form of the law of the iterated logarithm. It is therefore of interest to make the same kind of investigations regarding S_n^*. The problem is more difficult, since we have as yet no standard tools as in the case of S_n. We shall prove in this direction, as consequences of a more general but less handy inequality (Lemma 7), two theorems corresponding to the two types of estimation mentioned above. In order to state them we introduce the following notations. Let $E(X)$ denote the mathematical expectation of X. We shall assume that for each X, the first moment is zero, and the third absolute moment is finite. Thus we can write

$$(3) \qquad E(X_\nu) = 0;$$

$$(4) \qquad E(X_\nu^2) = \sigma_\nu^2; \qquad s_n^2 = \sum_{\nu=1}^{n} \sigma_\nu^2;$$

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(2) Numbers in brackets refer to the references cited at the end of the paper.

$$E(|X_\nu|^3) = \gamma_\nu; \qquad \Gamma_n = \sum_{\nu=1}^{n} \gamma_\nu. \tag{5}$$

Naturally we assume that $s_n \to \infty$. We shall further make the following assumption:

$$\max_{1 \leq \nu \leq n} \gamma_\nu \sigma_\nu^{-2} = O(s_n^{1-\theta}) \tag{6}$$

where θ is a fixed but arbitrarily small positive number. Then we can prove the following two theorems.

THEOREM 1. *If c is a positive constant, then we have*

$$\begin{aligned} \Pr(S_n^* < cs_n) = {} & \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp\left(-\frac{(2i+1)^2\pi^2}{8c^2}\right) \\ & + O\left(\left(\frac{\lg_2 s_n}{\lg s_n}\right)^{1/2}\right). \end{aligned} \tag{7}$$

THEOREM 2. If $g_n \downarrow 0$ *and*

$$g_n^{-1} = O((\lg_2 s_n)^{1/2}) \tag{8}$$

then we have(3)

$$\Pr(S_n^* < g_n s_n) = (1 + o(1)) \exp\left(-\frac{\pi^2}{8g_n^2}\right). \tag{9}$$

Theorem 2 is one of a number of possible statements; we give prominence to it here because it furnishes the means of proving the next group of theorems which we now consider.

We might attempt to extend the law of the iterated logarithm to S_n^*. This turns out to be illusory since the same law holds for S_n^* as for S_n. More precisely, if $\phi_n \uparrow \infty$, we have always ("i. o." standing for "infinitely often")

$$\Pr(S_n^* > \phi_n s_n \text{ i. o.}) = \Pr(S_n > \phi_n s_n \text{ i. o.}).$$

This is obvious since both S_n^* and $\phi_n s_n$ are monotone increasing functions of n. Hence in particular three of Feller's theorems [11] read as follows:

I. *If* $\sup |X_n| = O(s_n(\lg_2 s_n)^{-3/2})$ *and* $\phi_n^2 = 2 \lg_2 s_n + 3 \lg_3 s_n + 2 \lg_4 s_n + \cdots + 2 \lg_{p-1} s_n + (2+\delta) \lg_p s_n$ *then the probability*

$$\Pr(S_n^* > s_n \phi_n \text{ i. o.}) \tag{10}$$

(3) *Added in proof.* For the application of Theorem 2 in Lemma 9 it is important to notice that the constant in the $o(1)$ term in (9) depends only on the constants in the $O(1)$ terms in (6) and (8), and the θ in (6), but otherwise is independent of the random variables considered.

is equal to zero or one according as δ is positive or not.

II. *If $\phi_n \uparrow \infty$ and*

$$\sup |X_n| = O\left(\frac{s_n}{\phi_n^3}\right),$$

then (10) *is equal to zero or one according as the series*

$$\sum_n \frac{\sigma_n^2}{s_n^2} \phi_n e^{-(1/2)\phi_n^2}$$

is convergent or divergent.

III. *If $\phi(t) \uparrow \infty$ and*

$$\sup |X_n| = O\left(\frac{s_n}{\phi^3(s_n^2)}\right),$$

then $\Pr(S_n^* > s_n\phi(s_n^2) \text{ i. o.})$ *is equal to zero or one according as the integral*

$$\int^\infty \frac{1}{t}\phi(t)e^{-(1/2)\phi^2(t)}dt$$

is convergent or divergent.

These results give very precise upper bounds for S_n^*, with probability one. The question naturally arises as to the precise lower bounds for S_n^*. (We may mention that the analogous problem for S_n has been treated by Erdös and the author [4] and is radically different.) In this connection Erdös has communicated to the author the following result: there exist two constants $c_2 > c_1 > 0$ such that

$$\Pr\left(c_1 < \liminf \frac{S_n^*}{s_n(\lg_2 s_n)^{-1/2}} < c_2\right) = 1.$$

His method, of an elementary nature, does not seem capable of a sharper result. Using Theorem 2 stated above we can easily prove that

$$\Pr\left(\liminf \frac{S_n^*}{s_n(\lg_2 s_n)^{-1/2}} = 8^{-1/2}\pi\right) = 1.$$

This corresponds to Khintchine-Kolmogoroff's original form of the law of the iterated logarithm ([14] and [16]). However, we can go much further and prove the following theorems which are the exact counterparts of Feller's theorems cited above.

THEOREM 3. *Under the assumptions* (3) *to* (6), *if*

(11) $\phi_n^2 = \lg_2 s_n + 2 \lg_3 s_n + \lg_4 s_n + \cdots + \lg_{p-1} s_n + (1 + \delta) \lg_p s_n,$

then

$$\Pr (S_n^* < 8^{-1/2}\pi\phi_n^{-1} s_n \text{ i. o.}) \tag{12}$$

is equal to zero or one according as δ is positive or not.

THEOREM 4. *Under the same assumptions, if $\phi_n \uparrow \infty$, then* (12) *is equal to zero or one according as the series*

$$\sum \frac{\sigma_n^2}{s_n^2} \phi_n^2 e^{-\phi_n^2} \tag{13}$$

is convergent or divergent.

Theorem 3 is a particular case of Theorem 4.

THEOREM 5. *Under the same assumptions, if $\phi(s_n^2) \uparrow \infty$, then*

$$\Pr (S_n^* < 8^{-1/2}\pi\phi^{-1}(s_n^2)s_n \text{ i. o.}) \tag{14}$$

is equal to zero or one according as the integral

$$\int^\infty \frac{1}{t} \phi^2(t)e^{-\phi^2(t)}dt \tag{15}$$

is convergent or divergent.

The similarity between these theorems and Feller's is indeed striking. It should however be noted that the condition (6) is not the best possible, although it is weaker than those considered by Cramér [5]. That condition (6) can be trivially weakened will be apparent from the proof. But no complete settlement of the question seems in sight.

We outline the methods of proof as follows. We approximate the distribution on S_n^* by that of

$$\max_{1 \leq j \leq k} |S_{n_j}| \tag{16}$$

where k is an integer to be determined later and $0 < n_1 < \cdots < n_k = n$ is a suitably chosen sequence such that $s_{n_j}^2 \sim jk^{-1}s_n^2$.

In §2 we study the approximate distribution of (16). It is found to approach that of a k-dimensional normal distribution with a remainder we shall estimate. The treatment in Lemma 2(4), much to be preferred to the

(4) In the special case of equal components Bergström's result [2] seems to imply a better estimate than Lemma 2, replacing the factor 4^k by a fixed power of k. The improvement however is annulled by Lemma 3. It becomes essential in the problem of max S_ν, without the absolute value. We shall consider this elsewhere.

author's original proof using characteristic functions, is due to G. A. Hunt.

In §3 we estimate the difference between the distribution of S_n^* and that of (16). This is done by a substantial improvement of the method of Erdös and Kac (8), using sharper estimates resulting from the one-dimensional Berry-Esseen estimate. To obtain the approximate distribution of S_n^* it remains to evaluate the k-dimensional normal distribution obtained in §2. The problem appears to be one of multiple integrals but has not been worked out directly. Instead we use a quantitative refinement of the "invariance principle" of Erdös and Kac and study the simplest case of random walk. This latter problem, being almost classical, has been treated by many authors with different methods. However as we require not only the limiting distribution but also a remainder no reference seems available in the literature. We shall obtain the precise result by going back to a combinatorial formula due (apparently) to Bachelier [1]. After this we combine the results in §§2 and 3 to establish a theorem (Lemma 7) which includes Theorems 1 and 2 as particular cases.

In §4 we prove Theorems 3, 4 and 5. The proof of these theorems depends essentially on Theorem 2, which plays the role here as the theorem of Cramér-Feller does in the case of Feller's theorems cited above. Several arguments of Feller's are also used and the author's indebtedness to his previous work is considerable.

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2. **An approximation theorem for a certain multi-dimensional distribution.** We shall use $A_1, A_2, \cdots$ to denote absolute constants.

Let $n_1 < \cdots < n_k = n$ be a subsequence of $1, \cdots, n$ defined by the following:

$$s_{n_j}^2 \leqq j k^{-1} s_n^2 < s_{n_j+1}^2, \qquad j = 1, \cdots, k. \tag{17}$$

Then $(S_{n_1}, \cdots, S_{n_j})$ is a random point in j-dimensional space. Let its distribution function be

$$F_j(u_1, \cdots, u_j) = \Pr (S_{n_1} \leqq u_1, \cdots, S_{n_j} \leqq u_j). \tag{18}$$

Write also

$$F_j^*(x) = \Pr (S_{n_j} - S_{n_{j-1}} \leqq x), \qquad S_{n_0} = 0.$$

We put

$$B_j^2 = s_{n_j}^2 - s_{n_{j-1}}^2; \tag{19}$$

(20) $$M_n = \max_{1 \leqq \nu \leqq n} \gamma_\nu \sigma_\nu^{-2}.$$

LEMMA 1. *We have*

$$F_j^*(x) = \Phi_j^*(x) + R_j^*(x),$$

where $\Phi_j^*(x)$ *is the normal distribution function with mean* 0 *and variance* B_j^2, *and* $|R_j^*(x)| \leqq A_1 M_n B_j^{-1}$.

This is a restatement of the Berry-Esseen theorem.

LEMMA 2. *Suppose that* (6) *holds and also that*

(21) $$\max_{1 \leqq \nu \leqq n} \sigma_\nu^2 = o(k^{-1} s_n).$$

Then we have

(22) $$|F_j(u_1, \cdots, u_j) - \Phi_j(u_1, \cdots, u_j)| \leqq A_2 k^{1/2} 4^j M_n s_n^{-1},$$

where $\Phi_j(u_1, \cdots, u_j)$ *is the* j*-dimensional normal distribution function with the same moments of the first and second order as* $F_j(u_1, \cdots, u_j)$.

Proof. From (17), (19) and (21) it is easy to see that

(23) $$B_j \sim k^{-1/2} s_n.$$

Hence by Lemma 1, we have

(24) $$|R_j^*(x)| \leqq A_2 k^{1/2} M_n s_n^{-1}.$$

For $j=1$, $R_1(x) = R_1^*(x)$; hence (22) is true for $j=1$. Now we use induction on j. Assume that

(25) $$|R_j(u_1, \cdots, u_j)| \leqq A_2 k^{1/2} 4^j M_n s_n^{-1}.$$

We have, by the definition (18),

(26) $$\begin{aligned} F_{j+1}(u_1, \cdots, u_{j+1}) &= \int_{-\infty}^{\infty} F_j(u_1, \cdots, u_{j-1}, \min(u_j, u_{j+1} - x)) dF_{j+1}^*(x) \\ &= \int_{-\infty}^{\infty} \{\Phi_j(u_1, \cdots, u_{j-1}, \min(u_j, u_{j+1} - x)) \\ &\quad + R_j(u_1, \cdots, u_{j-1}, \min(u_j, u_{j+1} - x))\} d[\Phi_{j+1}^*(x) + R_{j+1}^*(x)] \\ &= \Phi_{j+1}(u_1, \cdots, u_{j+1}) + \int R_j d\Phi_{j+1}^* \\ &\quad + \int R_j d\Phi_{j+1}^* + \int \Phi_j dR_{j+1}^*. \end{aligned}$$

Evidently we have

$$\left|\int R_j d\Phi^*_{j+1}\right| \leqq \sup |R_j|,$$

$$\left|\int R_j dR^*_{j+1}\right| \leqq 2 \sup |R_j|.$$

Finally, using integration by parts, we have

$$\int_{-\infty}^{\infty} \Phi_j(u_1, \cdots, u_{j-1}, \min(u_j, u_{j+1} - x))dR^*_{j+1}(x)$$
$$= \int_{-\infty}^{u_{j+1}-u_j} \Phi_j(u_1, \cdots, u_j)dR^*_{j+1}(x)$$
$$+ \int_{u_{j+1}-u_j}^{\infty} \Phi_j(u_1, \cdots, u_j, u_{j+1} - x)dR^*_{j+1}(x)$$
$$= \Phi_j(u_1, \cdots, u_j)R^*_{j+1}(u_{j+1} - u_j) - \Phi_j(u_1, \cdots, u_j)R^*_{j+1}(u_{j+1} - u_j)$$
$$+ \int_{u_{j+1}-u_j}^{\infty} R^*_{j+1}(x)d\Phi_j(u_1, \cdots, u_{j-1}, u_{j+1} - x).$$

Hence the absolute value of the left-hand side is less than or equal to

$$\sup |R^*_{j+1}|.$$

Substituting these estimates into (22), we obtain

$$|F_{j+1} - \Phi_{j+1}| \leqq 3(\sup |R_j| + \sup |R^*_{j+1}|).$$

From (25) and (26), we have

$$|F_{j+1} - \Phi_{j+1}| \leqq 3A_2 k^{1/2} M_n s_n^{-1}(4^j + 1)$$
$$\leqq A_2 4^{j+1} k^{1/2} M_n s_n^{-1}.$$

Thus the induction is complete.

Now we put, for non-negative u_j's,

$$F_0(u_1, \cdots, u_k) = \Pr(|S_{n_1}| \leqq s_n u_1, \cdots, |S_{n_k}| \leqq s_n u_k), \tag{27}$$

$$\Phi_0(u_1, \cdots, u_k) = \frac{s_n^k}{(2\pi)^{k/2} B_1 \cdots B_k} \int_{-u_1}^{u_1} \cdots \int_{-u_k}^{u_k} \cdot \exp\left(-\frac{1}{2}\sum_{j=1}^{k} \frac{s_n^2}{B_j^2}(x_j - x_{j-1})^2\right) dx_1 \cdots dx_k. \tag{28}$$

LEMMA 3. *Under the same assumptions as in Lemma 2, we have*

$$(29) \qquad |F_0 - \Phi_0| \leqq A_3(10)^k M_n s_n^{-1}.$$

Proof. Taking $j=k$ in (22), we have

$$(30) \qquad |F_k - \Phi_k| \leqq A_2 k^{1/2} 4^k M_n s_n^{-1} \leqq A_3 5^k M_n s_n^{-1}.$$

It is well known that we have

$$\begin{aligned} F_0(u_1, \cdots, u_k) = F_k(s_n u_1, \cdots, s_n u_k) \\ - F_k(-s_n u_1, s_n u_2, \cdots, s_n u_k) - \cdots \\ - F_k(s_n u_1, \cdots, s_n u_{k-1}, -s_n u_k) \\ + F_k(-s_n u_1, -s_n u_2, s_n u_3, \cdots, s_n u_k) + \cdots \\ + (-1)^k F_k(-s_n u_1, \cdots, -s_n u_k); \end{aligned}$$

and a similar relation holds between Φ_k and Φ_0. Since there are 2^k terms on the right-hand side, (29) follows immediately from (30). It is not hard to obtain the explicit form of $\Phi_0(u_1, \cdots, u_k)$ in (28) by considering the covariance matrix.

3. **The distribution of the maximum partial sum.** Let c be a positive constant; g_n a monotone function of n; $\epsilon_n = o(1)$.

LEMMA 4. *Suppose that* (6) *and* (21) *are satisfied, and also that we have*

$$(31) \qquad \epsilon_n^2 g_n^2 = o(k^{-3/2} s_n^\theta),$$

$$(32) \qquad \sigma_n^2 = o((\epsilon_n^2 g_n^2 s_n^{3-\theta})^{2/3}).$$

Then we have

$$(33) \qquad \Pr(S_n^* < c g_n s_n) \geqq \Pr\left(\max_{1 \leqq j \leqq k} |S_{n_j}| < (c - \epsilon_n) g_n s_n\right) - R_n$$

where

$$(34) \qquad R_n \leqq A_4(k^{-1/2} \epsilon_n^{-1} g_n^{-1} \exp(-4^{-1} \epsilon_n^2 g_n^2 k) + (\epsilon_n g_n s_n^\theta)^{-2/3}).$$

Proof. Write

$$P_n = \Pr(S_n^* < c g_n s_n),$$

$$W_r = \Pr(S_{r-1}^* < c g_n s_n, |S_r| \geqq c g_n s_n).$$

Then we have

$$(35) \qquad \sum_{r=1}^n W_r = \Pr(S_n^* \geqq c g_n s_n) = 1 - P_n \leqq 1.$$

Suppose that $n_j < \gamma \leqq n_{j+1}$. We have

$$W_r = \Pr(S^*_{r-1} < cg_ns_n,\ |S_r| \geqq cg_ns_n) \Pr(|S_{n_{j+1}} - S_r| \geqq \epsilon_ng_ns_n) \tag{36}$$
$$+ \Pr(S^*_{r-1} < cg_ns_n,\ |S_r| \geqq cg_ns_n,\ |S_{n_{j+1}} - S_r| < \epsilon_ng_ns_n).$$

Let $A>0$ be an integer to be determined later. If $n_{j+1}-r \leqq A$, we have by the Tchebychef inequality,

$$\Pr(|S_{n_{j+1}} - S_r| \geqq \epsilon_ng_ns_n) \leqq (s^2_{n_{j+1}} - s^2_{n_{j+1}-A+1})(\epsilon_ng_ns_n)^{-2}. \tag{37}$$

If $n_{j+1}-r=B>A$, we have by the Berry-Esseen theorem,

$$\Pr(|S_{n_{j+1}} - S_r| \geqq \epsilon_ng_ns_n) = \left(\frac{2}{\pi}\right)^{1/2}\int_v^\infty e^{-u^2/2}du + O(\rho) \tag{38}$$

where

$$v = \epsilon_ng_ns_n(s^2_{n_{j+1}} - s^2_{n_{j+1}-A+1})^{-1/2}$$

and

$$\rho = M_n\left(\sum_{\nu=n_{j+1}-A+1}^{n_{j+1}} \sigma^2_\nu\right)^{-1/2}.$$

Hence from (38), since $A<B\leqq n_{j+1}-n_j$, $s^2_{n_{j+1}}-s^2_{n_{j+1}-A+1}\leqq s^2_{n_{j+1}}-s^2_{n_j}=B^2_{j+1}$,

$$\Pr(|S_{n_{j+1}} - S_r| \geqq \epsilon_ng_ns_n)$$
$$\leqq A_5\left(\frac{B_{j+1}}{\epsilon_ng_ns_n}\exp\left(-\frac{\epsilon^2_ng^2_ns^2_n}{2B^2_{j+1}}\right) + \frac{M_n}{(s^2_{n_{j+1}} - s^2_{n_{j+1}-A+1})^{1/2}}\right). \tag{39}$$

We choose A such that

$$\frac{M_n}{(s^2_{n_{j+1}} - s^2_{n_{j+1}-A+1})^{1/2}} \sim \frac{s^2_{n_{j+1}} - s^2_{n_{j+1}-A+1}}{\epsilon^2_ng^2_ns^2_n},$$

that is,

$$(s^2_{n_{j+1}} - s^2_{n_{j+1}-A+1})^{3/2} \sim \epsilon^2_ng^2_ns^2_nM_n.$$

Since $n_{j+1}-A+1\geqq n_j$ this is possible if, for example,

$$\epsilon^2_ng^2_ns^2_nM_n = o(B^3_{j+1}) = o(k^{-3/2}s^3_n)$$

by (23), and also if

$$\sigma^2_n = o((\epsilon^2_ng^2_ns^2_nM_n)^{2/3}).$$

These are implied by the conditions (31) and (32), on account of (6). Hence we obtain from (37) and (39)

$$\Pr(|S_{n_{j+1}} - S_r| \geqq \epsilon_n g_n s_n) \leqq A_5\left(\frac{B_{j+1}}{\epsilon_n g_n s_n} \exp\left(-\frac{\epsilon_n^2 g_n^2 s_n^2}{2B_{j+1}^2}\right) + (\epsilon_n g_n s_n^\theta)^{-2/3}\right).$$

Since $B_{j+1}^2 s_n^{-2} \sim k^{-1}$ by (23), we obtain

$$\Pr(|S_{n_{j+1}} - S_r| \geqq \epsilon_n g_n s_n) \leqq A_4\left(\frac{1}{k^{1/2}\epsilon_n g_n} \exp\left(-\frac{\epsilon_n^2 g_n^2 k}{4}\right) + (\epsilon_n g_n s_n^\theta)^{-2/3}\right).$$

If we denote the maximum of the left-hand side of this inequality for all r by R_n, (36) becomes

$$W_r \leqq R_n + \Pr(S_{r-1}^* < cg_n s_n,\ |S_r| \geqq cg_n s_n,\ |S_{n_{j+1}} - S_r| < \epsilon_n g_n s_n). \tag{40}$$

From (35) and (40), we obtain

$$\begin{aligned} &\Pr(S_n^* \geqq cg_n s_n) \\ &\quad \leqq R_n + \sum_{j=0}^{k-1} \sum_{r=n_j+1}^{n_{j+1}} \Pr(S_{r-1}^* < cg_n s_n,\ |S_r| \geqq cg_n s_n,\ |S_{n_{j+1}} - S_r| < \epsilon_n g_n s_n) \\ &\quad \leqq R_n + \Pr\left(\max_{1 \leqq j \leqq k} |S_{n_j}| \geqq (c - \epsilon_n) g_n s_n\right). \end{aligned}$$

This is equivalent to (33).

If in the function $F_0(u_1, \cdots, u_k)$ of (27) all the arguments are equal to u we shall use the shorter notation $F_{0k}(u)$; similarly for Φ_{0k}.

Lemma 5. *Suppose that the condition* (6) *is satisfied, and also for a* $\Theta > 0$ *we have*

$$\frac{8 \lg 10 \cdot \Theta}{\theta} \frac{\lg_2 s_n}{\lg s_n} \leqq \epsilon_n^2 g_n^2 = o\left(\frac{s_n^\theta}{(\lg s_n)^{3/2}}\right). \tag{41}$$

Then if we choose

$$k \sim \frac{\theta \lg s_n}{2 \lg 10} \tag{42}$$

we have

$$\Phi_{0k}((c - \epsilon_n) g_n) - L_n \leqq \Pr(S_n^* < cg_n s_n) \leqq \Phi_{0k}(cg_n) + L_n \tag{43}$$

where

$$L_n = O((\lg s_n)^{-\Theta}) \quad . \tag{44}$$

Proof. From (6) it follows that

$$\sigma_n \leqq \gamma_n \sigma_n^{-2} \equiv O(s_n^{1-\theta}).$$

Hence with the k in (42) condition (21) is satisfied. Further condition (32) in Lemma 4 is satisfied with $\epsilon_n^2 g_n^2$ satisfying (41). Condition (31) is clearly satisfied with (41) and the choice of k in (42). Hence both Lemma 3 and Lemma 4 are applicable. Taking all the u's in (29) to be $(c-\epsilon_n)g_n$ and recalling (27) we obtain

$$\begin{aligned}\Pr(S_n^* < cg_n s_n) &\geqq F_{0k}((c-\epsilon_n)g_n) - R_n \\ &\geqq \Phi_{0k}((c-\epsilon_n)g_n) - A_3(10)^k s_n^{-1} M_n - R_n.\end{aligned} \tag{45}$$

On the other hand we have

$$\Pr(S_n^* < cg_n s_n) \leqq F_{0k}(cg_n) \leqq \Phi_{0k}(cg_n) + A_3(10)^k s_n^{-1} M_n. \tag{46}$$

We find from (42) and (46)

$$(10)^k = O(s_n^{\theta/2}), \qquad (10)^k s_n^{-1} M_n = O(s_n^{-\theta/2}),$$

$$k^{-1/2} \epsilon_n^{-1} g_n^{-1} \exp(-4^{-1} \epsilon_n^2 g_n^2 k) = O((\lg s_n)^{-\Theta}),$$

$$(\epsilon_n g_n s_n^{\theta})^{-2/3} = O(s_n^{-\theta/2}).$$

Hence if we take

$$L_n = R_n + A_3(10)^k s_n^{-1} M_n = O((\lg s_n)^{-\Theta})$$

(45) and (46) imply (43).

LEMMA 6. *Suppose that for each ν,*

$$X_\nu = \begin{cases} +1 & \text{with probability } 1/2, \\ -1 & \text{with probability } 1/2. \end{cases} \tag{47}$$

Then if $g_n = o(n^{1/2})$, we have

$$\Pr(S_n^* < cg_n n^{1/2}) = T(cg_n) + O(g_n^{-1} n^{-1/2}) + O(n^{-1/2}) \tag{48}$$

where $T(x)$ is the distribution function defined by

$$T(x) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp\left(-\frac{(2i+1)^2\pi^2}{8x^2}\right), \qquad x > 0. \tag{49}$$

Proof. Write, for integral a and b,

$$P(a) = \Pr(S_n = a, -b < S_\nu < b, \text{ for } 0 < \nu \leqq n).$$

By a formula due to Bachelier [1, pp. 252–253],

$$2^n P(a) = C_{n,(n+a)/2} + \sum_{1 \leqq i \leqq (n+a)/2b} (-1)^i C_{n,(n+a)/2-ib} + \sum_{1 \leqq i \leqq (n-a)/2b} (-1)^i C_{n,(n-a)/2-ib}$$

if n and a have the same parity, otherwise $P(a) = 0$. Without loss of generality we may assume n to be even, b odd. Then

$$\sum_{-b<a<b, a\equiv 0 \pmod 2} C_{n,(n+a)/2-ib} = \sum_{(-b+1)/2 \leqq j \leqq (b-1)/2} C_{n,n/2+j-ib}.$$

Write

$$P_i = P_{-i} = \sum_{(-b+1)/2 \leqq j \leqq (b-1)/2} C_{n,n/2+j-ib} \frac{1}{2^n} = \sum_{n/2+1/2+(n^{1/2}/2)\zeta_{1i} \leqq m \leqq n/2-1/2+(n^{1/2}/2)\zeta_{2i}} C_{n,m};$$

where

$$\zeta_{1i} = -(2i+1)bn^{-1/2}, \qquad \zeta_{2i} = -(2i-1)bn^{-1/2}.$$

Finally we write

$$P = \sum_{-b<a<b} P(a). \tag{50}$$

From a formula of Uspensky [16, p. 129], noticing that the limits of the range of m are integers, we deduce easily that

$$\sum_{n/2+1/2+(n^{1/2}/2)\zeta_1 \leqq m \leqq n/2-1/2+(n^{1/2}/2)\zeta_2} C_{n,m} \frac{1}{2^n} = \left(\frac{1}{2\pi}\right)^{1/2} \int_{\zeta_1}^{\zeta_2} e^{-u^2/2} du + O\left(\frac{1}{n}\right).$$

Hence

$$P_{-i} + P_i = \left(\frac{2}{\pi}\right)^{1/2} \int_{\zeta_{1i}}^{\zeta_{2i}} e^{-u^2/2} du + O\left(\frac{1}{n}\right). \tag{51}$$

Since $-b<a<b$,

$$\frac{1}{2^n}\left| \sum_{1 \leqq i \leqq (n+a)/2b} (-1)^i C_{n,(n+a)/2-ib} + \sum_{1 \leqq i \leqq (n-a)/2b} (-1)^i C_{n,(n-a)/2-ib} - \sum_{1 \leqq i \leqq n/2b} (-1)^i \left[C_{n,(n+a)/2-ib} + C_{n,(n-a)/2-ib}\right] \right| \leqq \frac{1}{2^n}.$$

Hence

$$\tag{52} \begin{aligned} &\left| \sum_{-b<a<b,\, a\equiv 0 \pmod 2} P(a) - \sum_{1\leqq i\leqq n/2b} (-1)^i (P_i + P_{-i}) \right| \\ &\qquad\qquad \leqq \sum_{-b<a<b,\, a\equiv 0 \pmod 2} \frac{1}{2^n} \leqq \frac{b}{2^n}. \end{aligned}$$

Therefore from (50) to (52),

$$\tag{53} \begin{aligned} P &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{\zeta_{10}}^{\zeta_{20}} e^{-u^2/2} du + O\left(\frac{1}{n}\right) \\ &\quad + \sum_{1\leqq i\leqq n/2b} (-1)^i \left\{ \left(\frac{1}{2\pi}\right)^{1/2} \int_{\zeta_{1i}}^{\zeta_{2i}} e^{-u^2/2} du + O\left(\frac{1}{n}\right) \right\} + O\left(\frac{b}{2^n}\right) \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{\zeta_{10}}^{\zeta_{20}} e^{-u^2/2} du + \sum_{1\leqq i\leqq n/2b} (-1)^i \left(\frac{2}{\pi}\right)^{1/2} \int_{\zeta_{1i}}^{\zeta_{2i}} e^{-u^2/2} du \\ &\quad + O\left(\frac{1}{n}\right) + O\left(\frac{1}{b}\right) + O\left(\frac{b}{2^n}\right). \end{aligned}$$

Since the terms are alternating in sign and decreasing in absolute value, we have

$$\tag{54} \begin{aligned} &\left| \sum_{i>n/2b} (-1)^i \left(\frac{2}{\pi}\right)^{1/2} \int_{\zeta_{1i}}^{\zeta_{2i}} e^{-u^2/2} du \right| \\ &\qquad\qquad \leqq \left(\frac{2}{\pi}\right)^{1/2} \int_{(n-b)n^{-1/2}}^{(n+b)n^{-1/2}} e^{-u^2/2} du = O(e^{-n/3}), \end{aligned}$$

if $b = o(n)$.

Hence if $b = o(n)$, we obtain from (53) and (54),

$$\tag{55} \begin{aligned} P &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{\zeta_{10}}^{\zeta_{20}} e^{-u^2/2} du + \sum_{i=1}^{\infty} (-1)^i \left(\frac{2}{\pi}\right)^{1/2} \int_{\zeta_{1i}}^{\zeta_{2i}} e^{-u^2/2} du + O\left(\frac{1}{b}\right) \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \sum_{i=-\infty}^{\infty} (-1)^i \int_{(2i-1)bn^{-1/2}}^{(2i+1)bn^{-1/2}} e^{-u^2/2} du + O\left(\frac{1}{b}\right). \end{aligned}$$

We shall now construct a function $h(x)$ with period 2α as follows:

$$h(x) = \begin{cases} 1 & \text{if } 0 < x < \alpha/2, \\ -1 & \text{if } \alpha/2 < x < \alpha; \end{cases}$$

$$h(x) = h(-x); \quad h(x) = h(x + 2\alpha).$$

It is easy to find that

$$h(x) = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \cos\left(\frac{2i+1}{\alpha} \pi x\right).$$

Taking α to be $2bn^{-1/2}$ in the above, we have

$$\left(\frac{1}{2\pi}\right)^{1/2}\sum_{i=-\infty}^{\infty}(-1)^i\int_{(2i-1)bn^{-1/2}}^{(2i+1)bn^{-1/2}} e^{-u^2/2}du = \left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} h(x)e^{-x^2/2}dx$$

$$(56) \qquad = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{-x^2/2}\cos\left(\frac{(2i+1)\pi n^{1/2}}{2b}x\right)dx$$

$$= \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\exp\left(-\frac{(2i+1)^2\pi^2 n}{8b^2}\right)$$

since

$$\left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{-x^2/2}\cos txdx = e^{-t^2/2}.$$

Therefore from (55) and (56) we obtain

$$(57) \qquad P = T(bn^{-1/2}) + O(b^{-1}).$$

Since by assumption $g_n = o(n^{1/2})$, $cg_n n^{1/2} = o(n)$; taking b successively to be the nearest odd integers to $cg_n n^{1/2}$ in (54) and observing that $T(bn^{-1/2}) - T(cg_n) = O(n^{-1/2})$ we obtain (48).

LEMMA 7. *Returning to the general case, we have, if* (6) *and* (41) *are satisfied,*

$$(58) \qquad T((c - \epsilon_n)g_n) - H_n \leqq \Pr(S_n^* < cg_n s_n) \leqq T((c + \epsilon_n)g_n) + H_n;$$

where $T(x)$ is defined in (49) *and*

$$(59) \qquad H_n = O((\lg s_n)^{-\Theta} + g_n^{-1} s_n^{-1}).$$

Proof. For the special case (47), we have according to the general notation (4) and (5),

$$\sigma_m^2 = 1, \qquad \gamma_m = 1, \qquad s_m^2 = m, \qquad M_m = 1.$$

Condition (6) is satisfied with $\theta = 1$. Hence by Lemma 5, if

$$\frac{8 \lg 10 \cdot \Theta \lg_2 m^{1/2}}{\theta \lg m^{1/2}} \leqq \epsilon_m'^2 g_m'^2 = o\left(\frac{m^{1/2}}{(\lg m)^{3/2}}\right)$$

we have from (43),

$$(60) \qquad \Phi_{1k}((c - \epsilon_m')g_m') - L_m \leqq \Pr(S_m^* < cg_m' m^{1/2}) \leqq \Phi_{1k}(cg_m') + L_m,$$

where, by (42), k is given by

$$k \sim \frac{\theta \lg m}{4 \lg 10},$$

and where Φ_{1k} is obtained from Φ_{0k} in (28) after we replace s_n^2 by m and B_j by B_j' defined according to (17) and (19) by

$$m_j \leqq jk^{-1}m < m_j + 1,\ B_j'^2 = m_j - m_{j-1},$$

and where

$$L_m = O((\lg m)^{-\Theta}).$$

On the other hand, by Lemma 6, we have for the special case in question

$$\text{(61)} \qquad \Pr\,(S_m^* < cg_m'm^{1/2}) = T(cg_m') + O(g_m'^{-1}m^{-1/2}) + O(m^{-1/2}).$$

Substituting from (58) into (60) we obtain

$$\text{(62)} \qquad \Phi_{1k}(cg_m') - L_m \leqq T(cg_m') + O(g_m'^{-1}m^{-1/2} + m^{-1/2}) \leqq \Phi_{1k}(cg_m') + L_m.$$

From (62) we deduce

$$\text{(63)} \qquad \Phi_{1k}((c - \epsilon_m')g_m') \geqq T((c - \epsilon_m')g_m') + O(g_m'^{-1}m^{-1/2} + m^{-1/2}) + O(L_m);$$

$$\text{(64)} \qquad \Phi_{1k}(cg_m') \leqq T((c + \epsilon_m')g_m') + O(g_m'^{-1}m^{-1/2} + m^{-1/2}) + O(L_m).$$

Now putting

$$m = [s_n^2], \qquad \epsilon_m' = \epsilon_n, \qquad g_m' = g_n,$$

we obtain from (63) and (64) the following: if

$$\frac{8 \lg 10 \cdot \Theta}{\theta}\, \frac{\lg_2 s_n}{\lg s_n} \leqq \epsilon_n^2 g_n^2 = o\left(\frac{s_n^\theta}{(\lg s_n)^{3/2}}\right)$$

then we have

$$\text{(65)} \qquad \begin{aligned} T((c - \epsilon_n)g_n) - K_n \leqq \Phi_{1k}((c - \epsilon_n)g_n) &\leqq \Phi_{1k}(cg_n) \\ &\leqq T((c + \epsilon_n)g_n) + K_n, \end{aligned}$$

where

$$\text{(42 bis)} \qquad k \sim \frac{\theta \lg s_n}{2 \lg 10},$$

$$\text{(66)} \qquad K_n = O((\lg s_n)^{-\Theta} + g_n^{-1}s_n^{-1}).$$

Writing $\lambda_j = s_n B_j^{-1}$, $\lambda_j' = [s_n^2]^{1/2}B_j'^{-1}$ we have from (28)

$$\Phi_0(u_1, \cdots, u_k) = \frac{\lambda_1 \cdots \lambda_k}{(2\pi)^{k/2}} \int_{-u_1}^{u_1} \cdots \int_{-u_k}^{u_k} \exp\left(-\frac{1}{2}\sum_{j=1}^{k} \lambda_j^2(x_j - x_{j+1})^2\right) dx_1 \cdots dx_k.$$

It is easy to verify that

$$\left|\frac{\partial \Phi_0}{\partial \lambda_j}\right| \leqq \frac{3}{2\lambda_j}.$$

Hence

$$(67) \qquad | \Phi_{0k} - \Phi_{1k} | \leqq \frac{3}{2} \sum_{j=1}^{k} \frac{1}{\lambda_j} | \lambda_j - \lambda_j' |.$$

Since $\sigma_n = O(s_n^{1-\theta})$, $B_j^2 = s_n^2 k^{-1} + O(s_n^{2-2\theta})$,

$$\lambda_j^2 = \frac{s_n^2}{B_j^2} = k\left(1 + O\left(\frac{k}{s_n^{2\theta}}\right)\right).$$

The same holds for $\lambda_j'^2$. Thus $|\lambda_j - \lambda_j'| = O(k^2 s_n^{-2\theta})$; and by (42 bis) and (67),

$$(68) \qquad | \Phi_{0k} - \Phi_{1k} | = O(s_n^{-\theta}).$$

Therefore from (65) we obtain

$$(69) \qquad \begin{aligned} T((c - \epsilon_n)g_n) - J_n \leqq \Phi_{0k}((c - \epsilon_n)g_n) &\leqq \Phi_{0k}(cg_n) \\ &\leqq T((c + \epsilon_n)g_n) + J_n, \end{aligned}$$

where k is given by (42 bis) and from (66) and (68) we have

$$(70) \qquad J_n = O(R_n) + O(s_n^{-\theta}) = O((\lg s_n)^{-\Theta} + g_n^{-1} s_n^{-1}).$$

Using (69) in (43), Lemma 5, we obtain

$$T((c - \epsilon_n)g_n) - H_n \leqq \Pr(S_n^* < cg_n s_n) \leqq T((c + \epsilon_n)g_n) + H_n$$

where $H_n = O(J_n) + O(L_n)$. Hence by (44) and (70) we have established (58) and (59).

Proof of Theorem 1. Taking $g_n = 1$ in Lemma 5, we get

$$(71) \qquad T(c - \epsilon_n) - H_n \leqq \Pr(S_n^* < cs_n) \leqq T(c + \epsilon_n) + H_n,$$

where

$$H_n = O((\lg s_n)^{-\Theta}).$$

Now we have, by the mean-value theorem,

$$\exp\left(\frac{-(2i+1)^2\pi^2}{8(c+\epsilon_n)^2}\right) - \exp\left(\frac{-(2i+1)^2\pi^2}{8c^2}\right) \leqq \frac{(2i+1)^2\pi^2}{4c^3} \exp\left(-\frac{(2i+1)^2\pi^2}{8(c+\epsilon_n)^2}\right)\epsilon_n.$$

Hence

$$T(c+\epsilon_n) - T(c) \leqq \frac{4}{\pi}\sum_{i=0}^{\infty} \frac{(2i+1)\pi^2}{4c^3} \exp\left(-\frac{(2i+1)^2\pi^2}{8(c+\epsilon_n)^2}\right)\epsilon_n = O(\epsilon_n).$$

Thus (71) becomes

$$\text{(72)} \qquad \Pr(S_n^* < cs_n) = T(c) + O(\epsilon_n) + O((\lg s_n)^{-\Theta}).$$

Choosing, for example, $\Theta = 1$ and

$$\epsilon_n^2 = \frac{8 \lg 10}{\theta} \frac{\lg_2 s_n}{\lg s_n}$$

which is permissible by (41), we obtain (7) from (72).

Proof of Theorem 2. We have

$$\frac{4}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right) - \frac{4}{3\pi}\exp\left(-\frac{9\pi^2}{8x^2}\right) \leqq T(x) \leqq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8x^2}\right).$$

Since $\epsilon_n \downarrow 0$, we have if $\epsilon_n < 4^{-1}$,

$$\begin{aligned} T((1+\epsilon_n)g_n) &\leqq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8g_n^2}(1+\epsilon_n)^{-2}\right) \\ \text{(73)} \qquad &\leqq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8g_n^2}(1-2\epsilon_n)\right) \\ &= \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8g_n^2}\right)\exp\left(\frac{\pi^2\epsilon_n}{4g_n^2}\right), \end{aligned}$$

$$\begin{aligned} T((1-\epsilon_n)g_n) &\geqq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8g_n^2}(1-\epsilon_n)^{-2}\right) - \frac{4}{3\pi}\exp\left(-\frac{9\pi^2}{8g_n^2}\right) \\ \text{(74)} \qquad &\geqq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8g_n^2}(1+4\epsilon_n)\right) - \frac{4}{3\pi}\exp\left(-\frac{9\pi^2}{8g_n^2}\right) \\ &\geqq \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8g_n^2}\right)\exp\left(-\frac{\pi^2\epsilon_n}{2g_n^2}\right) - \frac{4}{3\pi}\exp\left(-\frac{9\pi^2}{8g_n^2}\right). \end{aligned}$$

Choosing

$$\epsilon_n^2 = \frac{8\lg 10\cdot\Theta}{\theta}\frac{\lg_2 s_n}{g_n^2 \lg s_n}$$

then (41) is satisfied, and from (8),

$$\frac{\epsilon_n^2}{g_n^4} = \frac{8\lg 10\cdot\Theta}{\theta}\frac{\lg_2 s_n}{g_n^6\lg s_n} = o(1).$$

Hence we have

$$\exp\left(\frac{\pi^2\epsilon_n}{4g_n^2}\right) = 1 + o(1),$$

$$\exp\left(-\frac{\pi^2\epsilon_n}{2g_n^2}\right) = 1 + o(1).$$

Since $g_n \downarrow 0$, we have

$$\exp\left(-\frac{9\pi^2}{8g_n^2}\right) = o\left(\exp\left(-\frac{\pi^2}{8g_n^2}\right)\right).$$

Thus from (73) and (74), we obtain

$$\frac{4}{\pi}(1 + o(1)) \exp\left(-\frac{\pi^2}{8g_n^2}\right) \leqq T((1-\epsilon_n)g_n) \leqq T((1+\epsilon_n)g_n)$$
$$\leqq \frac{4}{\pi}(1 + o(1)) \exp\left(-\frac{\pi^2}{8g_n^2}\right).$$

Therefore (58) becomes

$$\Pr(S_n^* < g_n s_n) = \frac{4}{\pi}(1 + o(1)) \exp\left(-\frac{\pi^2}{8g_n^2}\right) + O((\lg s_n)^{-\Theta}).$$

Since we may choose Θ arbitrarily large, (9) follows on account of (8).

4. **Some strong limit theorems.** Since we shall deal with indices n, ν, k and so on, which ultimately tend to infinity, we shall often omit mention of this proviso. Thus, sometimes our statements are true only when the appropriate index is sufficiently large.

The condition (6) is assumed in this section. From (6) it follows:

$$\sigma_n = O(s_n^{1-\theta}), \qquad \theta > 0. \tag{75}$$

Let $\psi_n \uparrow \infty$, and

$$\psi_n = O((\lg_2 s_n)^{1/2}). \tag{76}$$

Taking $g_n = \psi_n^{-1}$ in Theorem 2, we have

$$A_6 e^{-\psi_n^2} \leqq \Pr(S_n^* < 8^{-1/2}\pi s_n \psi_n^{-1}) \leqq A_7 e^{-\psi_n^2}. \tag{9 bis}$$

We shall construct a subsequence $\{n_k\}$, $k = 1, 2, \cdots$, as follows. Take $a > 0$. Put $n_1 = 1$. Suppose that n_k is defined already, then since $s_n \uparrow$, there is a unique n_{k+1} such that

$$s_{n_{k+1}-1} \leqq s_{n_k}(1 + a\psi_{n_k}^{-2}) \leqq s_{n_{k+1}}.$$

Hence (for k sufficiently large)

$$s_{n_{k+1}-1}^2 \leqq s_{n_k}^2(1 + 3a\psi_{n_k}^{-2}).$$

By virtue of (75) and (76), we have

$$\begin{aligned} s_{n_{k+1}}^2 &\leqq s_{n_k}^2 + 3as_{n_k}^2\psi_{n_k}^{-2} + \sigma_{n_{k+1}}^2 \leqq s_{n_k}^2 + 4as_{n_{k+1}}^2\psi_{n_k}^{-2}; \\ s_{n_{k+1}}^2 &\leqq s_{n_k}^2(1 - 4a\psi_{n_k}^{-2})^{-1}. \end{aligned}$$

Thus there exists $b>a$ such that

$$s_{n_k}(1 + a\psi_{n_k}^{-2}) \leqq s_{n_{k+1}} \leqq s_{n_k}(1 + b\psi_{n_k}^{-2}). \tag{77}$$

For simplicity we shall write k' for n_k, s_k' for s_{n_k}, ψ_k' for ψ_{n_k}, and so on.

LEMMA 8. *Suppose that $\psi_n \uparrow \infty$ and (76) holds. Let $\{n_k\}$ be any sequence satisfying (77). Then if*

$$\sum_k e^{-\psi_{n_k}^2} < \infty \tag{78}$$

we have

$$\Pr (S_n^* < 8^{-1/2}\pi s_n\psi_n^{-1} \text{ i. o.}) = 0. \tag{79}$$

Proof. From (9 bis) we have

$$\Pr\left(S_{k'}^* < \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}}\right) \leqq A_7e^{-(\psi_k'^2-3b)}.$$

Hence by (78)

$$\sum_{k=1}^{\infty} \Pr\left(S_{k'}^* < \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}}\right) < \infty.$$

By the lemma of Borel-Cantelli (see, for example [13, pp. 26–27]), we conclude that

$$\Pr\left(S_{k'}^* < \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}} \text{ i. o.}\right) = 0,$$

that is,

$$\Pr\left(S_{k'}^* \geqq \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}} \text{ for all sufficiently large } k\right) = 1. \tag{80}$$

Now suppose that $n_k<n\leqq n_{k+1}$. Then if

$$S_{k'}^* \geqq \frac{\pi}{8^{1/2}} \frac{s_k'}{(\psi_k'^2 - 3b)^{1/2}}$$

we have by (77)

$$S_n^* \geqq S_{k'}^* \geqq \frac{\pi}{8^{1/2}} \frac{s'_{k+1}}{(\psi_k'^2 - 3b)^{1/2}} \frac{s'_k}{s'_{k+1}}$$

$$\geqq \frac{\pi}{8^{1/2}} s'_{k+1}(\psi_k'^2 - 3b)^{-1/2}(1 + b\psi_k'^{-2})^{-1}.$$

If k is sufficiently large, we have

$$S_n^* \geqq \frac{\pi}{8^{1/2}} \frac{s'_{k+1}}{\psi'_k} \geqq \frac{\pi}{8^{1/2}} \frac{s_n}{\psi_n}.$$

Thus (80) entails

$$\Pr\left(S_n^* > \frac{\pi}{8^{1/2}} \frac{s_n}{\psi_n} \text{ for all sufficiently large } n\right) = 1.$$

This is equivalent to (79).

LEMMA 9. *Suppose that $\psi_n \uparrow \infty$ and (76) holds. Let $\{n_k\}$ be any sequence satisfying (77). Then if*

$$\sum_k e^{-\psi_{n_k}^2} = \infty \tag{81}$$

we have

$$\Pr\left(S_n^* < \frac{\pi}{8^{1/2}} \frac{s_n}{\psi_n} \text{ i. o.}\right) = 1. \tag{82}$$

Proof. By (77), given $s'_{k_{\nu-1}}$, there is a unique ν such that

$$s'_{k_\nu} \leqq s'_{k_{\nu-1}} \psi'^3_{k_{\nu-1}} < s'_{k_\nu+1}. \tag{83}$$

From (9 bis) we have, if $c > 1/8$ is any constant,

$$\Pr\left(S_{k'}^* < \frac{\pi}{8^{1/2}} \frac{s'_k}{(\psi_k'^2 + 8C)^{1/2}}\right) \geqq A_6 e^{-(\psi_k'^2 + 8C)}.$$

Hence by (81) we have

$$\sum_{\nu=1}^{\infty} \sum_{k=k_\nu}^{k_{\nu+1}^{-1}} \Pr\left(S_{k'}^* < \frac{\pi}{8^{1/2}} \frac{s'_k}{(\psi_k'^2 + 8C)^{1/2}}\right) = \infty. \tag{84}$$

Let $\{\nu(r)\}$, $r = 1, 2, \cdots$, denote the subsequence of $\nu = 1, 2, \cdots$ for which

$$\psi^2(k'_{\nu(r)+1}) > \psi^2(k'_{\nu(r)-1}) + 1. \tag{85}$$

Then we have

$$\Pr\left(S^*_{k'} < \frac{\pi}{8^{1/2}} \frac{s'_k}{(\psi'_k + 8C)^{1/2}}\right) \leqq A_7 e^{-(\psi'^2_k+8C)} \leqq A_8 \psi^{-2}(k') e^{-\psi^2(k')/2}.$$

From (83),

$$\psi'^3_{k_\nu} \geqq \frac{s'_{k_{\nu+1}}}{s'_{k_\nu}} \geqq \prod_{k=k_\nu}^{k_{\nu+1}-1} \left(1 + \frac{a}{\psi'^2}\right) \geqq a \sum_{k=k_\nu}^{k_{\nu+1}-1} \frac{1}{\psi'^2}.$$

Hence

$$\sum_{k=k_{\nu(r)}}^{k_{\nu(r)+1}-1} \Pr\left(S^*_{k'} < \frac{\pi}{8^{1/2}} \frac{s_k{}'}{(\psi'^2_k + 8C)^{1/2}}\right) \leqq A_8 e^{-\psi^2(k'_{\nu(r)})/2} \sum_{k=k_{\nu(r)}}^{k_{\nu(r)+1}-1} \frac{}{\psi'^2_k}$$

$$\leqq \frac{A_8}{a} \psi'^3_{k_{\nu(r)}} e^{-\psi^2(k'_{\nu(r)})/2} \leqq A_9 e^{-\psi^2(k'_{\nu(r)})/4}.$$

Thus by (85)

$$\sum_r \sum_{k=k_{\nu(r)}}^{k_{\nu(r)+1}-1} \Pr\left(S^*_{k'} < \frac{\pi}{8^{1/2}} \frac{s'_k}{(\psi'^2_k + 8C)^{1/2}}\right) \leqq A_9 \sum_r e^{-\psi^2(k'_{\nu(r)})/4} < \infty. \tag{86}$$

From (82) and (86) we conclude that if we delete the values of ν equal to $\nu(r)$, $r=1, 2, \cdots$, in (84) the remaining series is still divergent. Without loss of generality we may then assume that, for some fixed ν_0,

$$\sum_{\nu=1, \nu\equiv\nu_0 \pmod 2}^{\infty}{}' \sum_{k=k_\nu}^{k_{\nu+1}-1} \Pr\left(S^*_{k'} < \frac{\pi}{8^{1/2}} \frac{s'_k}{(\psi'^2_k + 8C)^{1/2}}\right) = \infty, \tag{87}$$

where the prime after the summation indicates the omission of the values $\nu(r)$.

Denote by:

E_μ the event

$$S^*_\mu < \frac{\pi}{8^{1/2}} \frac{s_\mu}{\psi_\mu},$$

$E'_{\nu-1}$ the event

$$S^*_{k'_{\nu-1}} \leqq C s_{k'_{\nu-1}},$$

$E_{\nu-1,\mu}$ the event

$$\max_{k'_{\nu-1}<\rho\leqq\mu} |S_\rho - S_{k'_{\nu-1}}| < \frac{\pi}{8^{1/2}} \frac{s_\mu}{(\psi^2_\mu + 8C)^{1/2}}, \qquad k'_\nu \leqq \mu \leqq k'_{\nu+1}.$$

If $\nu \neq \nu(r)$, then from (83) and (77) we have, since $\psi'^2_{k_\nu} \leqq \psi'^2_{k_{\nu-1}} + 1$, $\psi'^3_{k_\nu} \leqq 2^{1/2} \psi'^3_{k_{\nu-1}}$ for large ν,

$$s'_{k_{\nu-1}} < \psi'^{-3}_{k_{\nu-1}} s_{k_\nu}(1 + b\psi'^{-2}_{k_\nu}) \leqq s'_{k_\nu} 2^{1/2} \psi'^{-3}_{k_\nu} (1 + b\psi'^{-2}_{k_\nu}) \leqq 2 s'_{k_\nu} \psi'^{-3}_{k_\nu}. \tag{88}$$

Then if we have the conjunction $E'_{\nu-1}E_{\nu-1,\mu}$, we have by (88)

$$S_\mu^* < \frac{\pi}{8^{1/2}} \frac{s_\mu}{(\psi_\mu^2 + 8C)^{1/2}} + Cs'_{k_{\nu-1}} < \frac{\pi}{8^{1/2}}\left(\frac{s_\mu}{(\psi_\mu^2 + 8C)^{1/2}} + \frac{2Cs_\mu}{\psi_\mu^3}\right)$$
$$< \frac{\pi}{8^{1/2}} \frac{s_\mu}{\psi_\mu}\left(\frac{1}{(1 + 8C\psi_\mu^{-2})^{1/2}} + \frac{2C}{\psi_\mu^2}\right) < \frac{\pi}{8^{1/2}} \frac{s_\mu}{\psi_\mu}.$$

Therefore if $\nu \neq \nu(r)$, the conjunction $E'_{\nu-1}E_{\nu-1,\mu}$ implies E_μ. Writing

$$F_\nu = \sum_{\mu=k'_\nu}^{k'_{\nu+1}-1} E_\mu, \qquad F'_\nu = \sum_{\mu=k'_\nu}^{k'_{\nu+1}-1} E_{\nu-1,\mu},$$

we have, if $\nu \neq \nu(r)$, $E'_{\nu-1}F'_\nu$ implies F_ν, hence

$$\sum_{\nu=\nu_1}^{\infty}{}' E'_{\nu-1}F'_\nu \quad \text{implies} \quad \sum_{\nu=\nu_1}^{\infty}{}' F_\nu,$$

$$\text{(89)} \qquad \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' E'_{\nu-1}F'_\nu\right) \leqq \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' F_\nu\right).$$

The events F'_ν, $F'_{\nu+2}$, $F'_{\nu+4}$, $\cdots$ are independent and F'_j for $j \geqq \nu$ is independent of $E'_{\nu-1}$. By the Kolmogoroff inequality [15] we have

$$\text{(90)} \qquad \Pr(E'_{\nu-1}) \geqq 1 - 1/C^2 \cdot$$

We obtain, by an obvious argument and (90), for all $\nu_1 \geqq \nu_0$,

$$\text{(91)} \qquad \begin{aligned} \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' E'_{\nu-1}F'_\nu\right) &= \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' (E'_{\nu-1}F'_\nu - E'_{\nu-1}F'_\nu \sum_{j=\nu+1}^{\infty} E'_{j-1}F'_j\right) \\ &\geqq \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' E'_{\nu-1}\left(F'_\nu - F'_\nu \sum_{j=\nu+1}^{\infty}{}' F'_j\right)\right) \\ &= \sum_{\nu=\nu_1}^{\infty}{}' \Pr(E'_{\nu-1}) \Pr\left(F'_\nu - F'_\nu \sum_{j=\nu+1}^{\infty}{}' F'_j\right) \\ &\geqq \left(1 - \frac{1}{C^2}\right) \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' F'_\nu\right). \end{aligned}$$

Hence by (89) and (91) we have

$$\text{(92)} \qquad \begin{aligned} \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' F_\nu\right) &\geqq \left(1 - \frac{1}{C^2}\right) \Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' F'_\nu\right) \\ &\geqq \left(1 - \frac{1}{C^2}\right) \Pr\left(\sum_{\nu=\nu_1, \nu \equiv \nu_0 \pmod 2}^{\infty}{}' F'_\nu\right). \end{aligned}$$

Since the events F'_{ν_0}, F'_{ν_0+2}, F'_{ν_0+4}, $\cdots$ are independent, by the lemma of

Borel-Cantelli we know that

$$\Pr\left(\sum_{\nu=\nu_1, \nu \equiv \nu_0 \pmod 2}^{\infty}{}' F'_\nu \right) = 1$$

if and only if

$$\sum_{\nu=1, \nu \equiv \nu_0 \pmod 2}^{\infty}{}' \Pr(F'_\nu) = \infty. \tag{93}$$

If we can prove (93), then from this remark and (91) we shall have for all $\nu_1 \geqq \nu_0$, hence in fact for all ν_1,

$$\Pr\left(\sum_{\nu=\nu_1}^{\infty}{}' F_\nu \right) \geqq 1 - \frac{1}{C^2},$$

a fortiori, for all n_1,

$$\Pr\left(\sum_{n=n_1}^{\infty} E_n \right) \geqq 1 - \frac{1}{C^2}.$$

Since we may choose C arbitrarily large while the left-hand side does not depend on C we shall have proved for all n_1, $\Pr(\sum_{n=n_1}^{\infty} E_n) = 1$, which is equivalent to (82).

Hence to prove (82) it is sufficient to prove (93). By definition this is equivalent to

$$\sum_{\nu=1, \nu \equiv \nu_0 \pmod 2}^{\infty}{}' \Pr\left(\sum_{\mu=k'_\nu}^{k'_{\nu+1}-1} E_{\nu-1,\mu} \right) = \infty. \tag{94}$$

Comparing (94) and (87) we see that in order to prove (94) it is sufficient to prove that for $\nu \neq \nu(r)$, there exists a constant $A_{10} > 0$ such that for all sufficiently large ν, the following shall hold:

$$\Pr\left(\sum_{\mu=k'_\nu}^{k'_{\nu+1}-1} E_{\nu-1,\mu} \right) \geqq A_{10} \sum_{k=k_\nu}^{k_{\nu+1}-1} \Pr\left(S^*_{k'} < \frac{\pi}{8^{1/2}} \frac{s_k{}'}{(\psi'^2_k + 8C)^{1/2}} \right). \tag{95}$$

We have for any integer $N > 0$,

$$\begin{aligned} \Pr\left(\sum_{\mu=k'_\nu}^{k'_{\nu+1}-1} E_{\nu-1,\mu} \right) &\geqq \Pr\left(\sum_{k=k_\nu}^{k_{\nu+1}-1} E_{\nu-1,k'} \right) \\ &\geqq \frac{1}{N} \sum_{k=k_\nu}^{k_{\nu+1}-1} \Pr\left(E_{\nu-1,k'} - E_{\nu-1,k'} \sum_{j=k+N}^{k_{\nu+1}} E_{\nu-1,j'} \right). \end{aligned} \tag{96}$$

Now we see easily that $E_{\nu-1,k'} E_{\nu-1,j'}$ implies $E'_{k,j}$ where $E'_{k,j}$ denotes the event

$$\max_{n_k<i\leqq n_j} |S_i - S_{n_k}| < \frac{\pi}{8^{1/2}}\left(\frac{s_{n_k}}{(\psi_{n_k}^2+8C)^{1/2}} + \frac{s_{n_j}}{(\psi_{n_j}^2+8C)^{1/2}}\right).$$

Since $E_{\nu-1,k'}$ and $E'_{k,j}$ are independent, we have

$$\Pr(E_{\nu-1,k'}E_{\nu-1,j'}) \leqq \Pr(E_{\nu-1,k'})\Pr(E'_{k,j}).$$

If we can prove that, for a suitable N,

$$\sum_{j=k+N}^{k_{\nu+1}} \Pr(E'_{k,j}) < \frac{1}{2}, \tag{97}$$

then from (96)

$$\begin{aligned}\Pr\left(\sum_{\mu=k'_\nu}^{k'_{\nu+1}} E_{\nu-1,\mu}\right) &\geqq \frac{1}{N}\sum_{k=k_\nu}^{k_{\nu+1}} \Pr\left(E_{\nu-1,k'} - \sum_{j=k+N}^{k_{\nu+1}} E_{\nu-1,k'}E_{\nu-1,j'}\right)\\ &\geqq \frac{1}{N}\sum_{k=k_\nu}^{k_{\nu+1}} \Pr(E_{\nu-1,k'})\left(1-\sum_{j=k+N}^{k_{\nu+1}}\Pr(E'_{k,j})\right)\\ &\geqq \frac{1}{2N}\sum_{k=k_\nu}^{k_{\nu+1}} \Pr(E_{\nu-1,k'}).\end{aligned} \tag{98}$$

By (9 bis)[4] we have

$$\begin{aligned}\Pr(E_{\nu-1,k'}) &\leqq \Pr\left(\max_{k'_{\nu-1}<\rho\leqq k'} |S_\rho - S_{k'_{\nu-1}}| < 8^{-1/2}\pi\frac{(s_{k'}^2 - s_{k'_{\nu-1}}^2)^{1/2}}{(\psi_{k'}^2+8C)^{1/2}}\right)\\ &\geqq \frac{A_6}{A_7}\Pr\left(S^*_{k'} < 8^{-1/2}\pi\frac{s_k'}{(\psi_{k'}^2+8C)^{1/2}}\right).\end{aligned}$$

Thus from (98) and the last inequality we shall have proved (95) with $A_{10}=A_6/2NA_7$. Hence it is sufficient to prove (97).

Now we have, since $\psi_k'^2 \geqq \psi_{k_\nu}'^2 \geqq \psi_{k_{\nu+1}}'^2 - 1 \geqq \psi_j'^2 - 1$,

$$\frac{s_k'}{(\psi_k'^2+8C)^{1/2}} < \frac{s_j'}{(\psi_j'^2-1+8C)^{1/2}},$$

$$\Pr(E'_{k,j}) \leqq \Pr\left(\max_{n_k\leqq i\leqq n_j} |S_i - S_{n_k}| < \frac{\pi}{8^{1/2}}(s_j'^2 - s_k'^2)^{1/2} g_j\right) \tag{99}$$

where

$$g_j = \frac{2s_j'}{(s_j'^2 - s_k'^2)^{1/2}(\psi_j'^2+8C-1)^{1/2}}. \tag{100}$$

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[4] See footnote 3.

It is obvious that $g_j \downarrow 0$; in order to apply Theorem 2 we have to verify that

$$\frac{(s_j'^2 - s_k'^2)^{1/2}(\psi_j'^2 + 8C - 1)^{1/2}}{s_j'} \leqq A_{11}(\lg_2 (s_j'^2 - s_k'^2))^{1/2}$$

which is evident since

$$\left(\frac{s_j'^2 - s_k'^2}{\lg_2 (s_j'^2 - s_k'^2)}\right)^{1/2} \leqq A_{12} \frac{s_j'}{\psi_j'},$$

by (76). Therefore we have from (99) and (9), Theorem 2,

(101) $$\Pr (E'_{k,j}) \leqq A_{13}e - g_j^{-2}.$$

We have for sufficiently large k, from (77),

(102) $$\frac{s_k'}{s_{k+1}'} \leqq 1 - \frac{a}{2\psi_k'^2}, \qquad \frac{s_k'}{s_j'} \leqq \left(1 - \frac{a}{2\psi_j'^2}\right)^{j-k}.$$

If $hx \leqq \delta$ where $\delta > 0$ is sufficiently small, then $(1-x)^h \leqq 1-\delta' hx$ where $\delta' > 0$ is another constant. Hence if $j-k \leqq \delta\psi_j'^2$ we have from (102)

$$\frac{s_k'}{s_j'} \leqq 1 - \frac{\delta' a(j-k)}{\psi_j'^2}, \qquad 1 - \frac{s_k'^2}{s_j'^2} \geqq a' \frac{j-k}{\psi_j'^2}$$

where $a' > 0$. Then from (100)

$$g_j \leqq 2(a'(j-k))^{-1/2}.$$

Hence by (101) we have

(103) $$\Pr (E'_{k,j}) \leqq A_{13} \exp (- 4^{-1}a'^2(j-k)).$$

If $hx > \delta$, then $(1-x)^h < \delta'' < 1$, hence from (102), if $j-k > \delta\psi_j'^2$,

$$\frac{s_k'}{s_j'} < \delta_0 < 1, \qquad 1 - \frac{s_k'^2}{s_j'^2} \geqq 1 - \delta_0^2;$$

$$g_j \leqq 2(1-\delta_0^2)^{-1/2}\psi_j'^{-1};$$

(104) $$\Pr (E'_{k,j}) \leqq A_{13} \exp (- 4^{-1}(1-\delta_0^2)\psi_j'^2).$$

From (103) and (104),

(105) $$\sum_{j=k+N}^{k_{\nu+1}} \Pr (E'_{k,j}) \leqq A_{13}\left(\sum_{i=N}^{\infty} e^{-a'i/4} + (k_{\nu+1} - k_\nu) \exp\left(-\frac{1-\delta_0^2}{4}\psi_k'^2\right)\right).$$

We have by (83),

$$\psi_{k_\nu}'^3 \geqq \frac{s_{k_{\nu+1}}'}{s_{k_\nu}'} \geqq \left(1 + \frac{a}{\psi_{k_{\nu+1}-1}'^2}\right) \cdots \left(1 + \frac{a}{\psi_{k_\nu}'^2}\right) \geqq 1 + \frac{a(k_{\nu+1} - k_\nu)}{\psi_{k_{\nu+1}}'^2}.$$

Hence we have

$$k_{\nu+1} - k_\nu \leqq A_{14}\psi'^5_{k_{\nu+1}}.$$

Since $\nu \neq \nu(r)$, we have $\psi'^2_{k_{\nu+1}} \leqq 2\psi'^2_{k_\nu}$. Hence

$$k_{\nu+1} - k_\nu \leqq 6A_{14}\psi'^5_{k_\nu};$$

$$(k_{\nu+1} - k_\nu) \exp\left(-\frac{1-\delta_0^2}{4}\psi_k'^2\right) \leqq A_{15}\psi'^5_{k_\nu}e^{-A_{16}\psi'^2_{k_\nu}} = o(1).$$

Thus by choosing N sufficiently large we obtain from (105) the desired (97), if ν is sufficiently large. The proof of Lemma 9 is thus complete.

LEMMA 10. *If* $\{n_k\}$, $k=1, 2, \cdots$, *is defined by* (77), *then the series*

$$\sum_k e^{-\psi^2_{n_k}}$$

and

$$\sum_n \frac{\sigma_n^2}{s_n^2}\psi_n^2 e^{-\psi_n^2}$$

converge and diverge together.

Proof. We have

$$\frac{\sigma_n^2}{s_n^2} = 1 - \frac{s_{n-1}^2}{s_n^2}.$$

Since $x \leqq -\lg(1-x) \leqq 2x$ if $0<x<1$, we obtain

$$\frac{\sigma_n^2}{s_n^2} \leqq -\lg\left(1 - \frac{\sigma_n^2}{s_n^2}\right) = \lg\frac{s_n^2}{s_{n-1}^2},$$

$$\sum_{n_k<n\leqq n_{k+1}} \frac{\sigma_n^2}{s_n^2} \leqq \lg\frac{s'^2_{k+1}}{s'^2_k} \leqq 2\lg\left(1+\frac{b}{\psi_k'^2}\right) \leqq \frac{2b}{\psi_k'^2},$$

$$\frac{2\sigma_n^2}{s_n^2} \geqq -\lg\left(1 - \frac{\sigma_n^2}{s_n^2}\right) = \lg\frac{s_n^2}{s_{n-1}^2},$$

$$2\sum_{n_k<n\leqq n_{k+1}} \frac{\sigma_n^2}{s_n^2} \geqq \lg\frac{s'^2_{k+1}}{s'^2_k} \geqq 2\lg\left(1+\frac{a}{\psi_k'^2}\right) \geqq \frac{a}{\psi_k'^2}.$$

Since $\psi_n^2 e^{-\psi_n^2}\downarrow$, we have

$$\frac{a}{2}e^{-\psi'^2_{k+1}} = \psi_k'^2 e^{-\psi'^2_{k+1}}\frac{a}{2\psi_k'^2} \leqq \sum_{n_k<n\leqq n_{k+1}}\frac{\sigma_n^2}{s_n^2}\psi_n^2 e^{-\psi_n^2} \leqq \psi_k'^2 e^{-\psi_k'^2}\frac{2b}{\psi_k'^2} \leqq 2be^{-\psi_k'^2}.$$

Lemma 10 follows from this inequality.

Proof of Theorem 3. The ϕ_n given in (11) is monotone increasing and $\phi_n = O((\lg_2 s_n)^{1/2})$. Hence Lemma 8 and Lemma 9 are applicable. Hence

$$\Pr (S_n^* < 8^{-1/2}\pi s_n \phi_n^{-1} \text{ i. o.}) = \begin{cases} 0 \\ 1 \end{cases}$$

according as

$$\sum_k e^{-\psi_{n_k}^2} \begin{Bmatrix} < \\ = \end{Bmatrix} \infty.$$

By Lemma 10, the last series converges and diverges with (13), which in this case is

$$\sum_n \frac{(1 + o(1))\sigma_n^2 \lg_2 s_n}{s_n^2 \lg s_n (\lg_2 s_n)^2 \lg_3 s_n \cdots \lg_p s_n (\lg_{p+1} s_n)^{1+\delta}}.$$

Hence a well known theorem of Abel-Dini asserts that this is convergent if and only if δ is positive. Thus Theorem 3 is proved.

Proof of Theorem 4. Suppose that $\phi_n \uparrow \infty$. Define

$$\psi_n^2 = \min (\phi_n^2, 2 \lg_2 s_n). \tag{106}$$

If (13) is convergent, then

$$\sum_n \frac{\sigma_n^2}{s_n^2} \psi_n^2 e^{-\psi_n^2} = \sum_{\psi_n = \phi_n} + \sum_{\psi_n^2 \geqq 2 \lg_2 s_n} < \infty$$

since again by the Abel-Dini theorem we have

$$\sum_{\psi_n^2 \geqq 2 \lg_2 s_n} \leqq \sum \frac{2\sigma_n^2 \lg_2 s_n}{s_n^2 (\lg s_n)^2} < \infty.$$

By the definition (106) ψ_n satisfies (76), hence by Lemma 8,

$$\Pr (S_n^* < 8^{-1/2}\pi s_n \psi_n^{-1} \text{ i. o.}) = 0.$$

Since $\psi_n \leqq \phi_n$, a fortiori (12) is equal to zero.

If (13) is divergent, then since $\psi_n \leqq \phi_n$, we have

$$\sum_n \frac{\sigma_n^2}{s_n^2} \psi_n^2 e^{-\psi_n^2} = \infty.$$

Since ψ_n satisfies (76), by Lemma 9, $\Pr (S_n^* < 8^{-1/2}\pi s_r \psi_n^{-1} \text{ i. o.}) = 1$. By Theorem 3, we have

$$\Pr \left(S_n^* < 8^{-1/2}\pi \frac{s_n}{(2 \lg_2 s_n)^{1/2}} \text{ i. o.} \right) = 0.$$

Hence there exists a subsequence n_i such that $\psi_{n_i}^2 \leqq 2 \lg_2 s_{n_i}$ and

$$\Pr\left(S^*_{n_i} < 8^{-1/2}\pi s_{n_i}\psi^{-1}_{n_i} \text{ i. o.}\right) = 1.$$

By the definition (106), we have $\psi_{n_i} = \phi_{n_i}$. Hence (12) is equal to one. Theorem 4 is proved.

Proof of Theorem 5. By Theorem 4 it is sufficient to prove that the series

$$\sum_n \frac{\sigma_n^2}{s_n^2}\phi^2(s_n^2)e^{-\phi^2(s_n^2)} \tag{107}$$

and the integral (15) converge and diverge together.

We have, since $t^{-1}\phi^2(t)e^{-\phi^2(t)} \downarrow 0$,

$$\int_{s_k^2}^{\infty} t^{-1}\phi^2(t)e^{-\phi^2(t)}dt = \sum_{n=k+1}^{\infty}\int_{s_{n-1}^2}^{s_n^2} t^{-1}\phi^2(t)e^{-\phi^2(t)}dt$$
$$\geqq \sum_{n=k+1}^{\infty} \frac{s_n^2 - s_{n-1}^2}{s_n^2}\phi^2(s_n^2)e^{-\phi^2(s_n^2)}.$$

Hence if (107) diverges, (15) diverges too.

On the other hand, we have

$$\sum_{n=N+1}^{\infty} \frac{\sigma_n^2}{s_{n-1}^2}\phi^2(s_{n-1}^2)e^{-\phi^2(s_{n-1}^2)} \geqq \int_{s_N^2}^{\infty} t^{-1}\phi^2(t)e^{-\phi^2(t)}dt. \tag{108}$$

From (75) we have $s_n^2 = s_{n-1}^2 + \sigma_n^2 \leqq s_{n-1}^2 + O(s_n^{2-2\theta})$. Hence if n is large enough, we have

$$s_n^2 \leqq 2s_{n-1}^2. \tag{109}$$

Let n_k, $k=1, 2, \cdots$, denote the subsequence of $n=1, 2, \cdots$ for which

$$\phi^2(s_{n_k}^2) > \phi^2(s_{n_k-1}^2) + 1. \tag{110}$$

Evidently we have by (109) and (110),

$$\sum_k \frac{\sigma_{n_k}^2}{s_{n_k-1}^2}\phi^2(s_{n_k-1}^2)e^{-\phi^2(s_{n_k-1}^2)} \leqq A_{16}\sum_k \frac{\sigma_{n_k}^2}{s_{n_k}^2} e^{-\phi^2(s_{n_k}^2-1)/2} < \infty. \tag{111}$$

Hence if (15) diverges, we have, by (108) and (111),

$$\sum_{n=1, n\neq n_k}^{\infty} \frac{\sigma_n^2}{s_{n-1}^2}\phi^2(s_{n-1}^2)e^{-\phi^2(s_{n-1}^2)} = \infty. \tag{112}$$

By (110) if $n \neq n_k$, we have $\phi^2(s_n^2) \leqq \phi^2(s_{n-1}^2)+1$. From this and (112) we obtain

$$\sum \frac{\sigma_n^2}{s_n^2}\phi^2(s_n^2)e^{-\phi^2(s_n^2)} \geqq \frac{e^{-1}}{2}\sum \frac{\sigma_n^2}{s_{n-1}^2}\phi^2(s_{n-1}^2)e^{-\phi^2(s_{n-1}^2)} = \infty.$$

Theorem 5 is proved.

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